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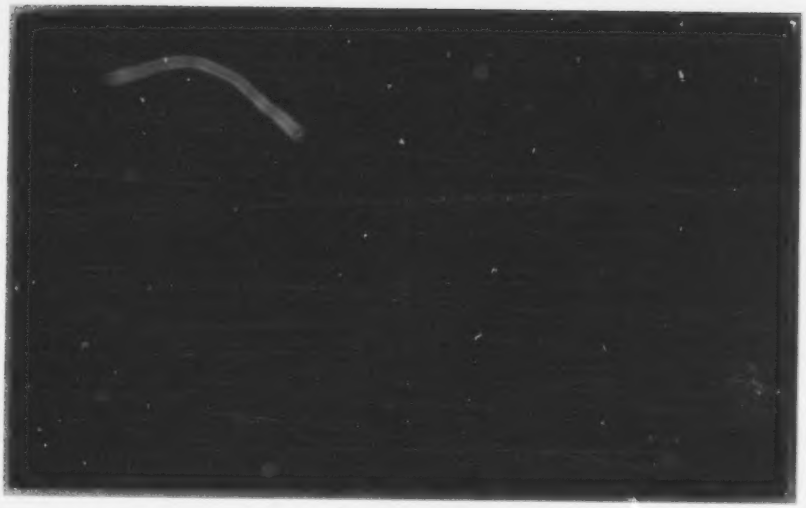

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THE VEHICLE ROUTING PROBLEM WITH STOCHASTIC TRAVEL TIMES

by

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ABSTRACT

This paper considers vehicle routing problems (VRPs) with stochastic service and travel times, in which vehicles incur a penalty proportional to the duration of their route in excess of a preset constant. Three mathematical programming models are presented: a chance constrained model, a three-index simple recourse model and a two-index recourse model. A general branch and cut algorithm for the three models is described. Computational results indicate that moderate size problems can be solved to optimality.

Key-words : stochastic vehicle routing problems, chance constrained models, recourse models.

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RÉSUMÉ

On considère dans cet article des problèmes de tournées avec temps de service et temps de voyage stochastiques, dans lesquels les tournées en excès d'une certaine valeur engendrent une pénalité proportionnelle à l'excédent. Trois modèles de programmation mathématique sont présentés : un modèle avec contraintes probabilistes, un modèle avec recours simple à trois indices ainsi qu'un modèle avec recours à deux indices. On procède ensuite à la description d'un algorithme de "branch and cut" applicable aux trois modèles. Les résultats numériques indiquent que l'approche proposée permet de résoudre de façon exacte des problèmes de taille moyenne.

Mots-clés : problèmes de tournées stochastiques, modèles probabilistes, modèles avec recours.

INTRODUCTION

The **Vehicle Routing Problem** (VRP) can be defined as follows. Let $G=(V_0,E)$ be a graph where $V_0 = \{v_0.v_1.....v_n\}$ is the **vertex set**; $V=V_0 \setminus \{v_0\}$. and $E = \{(v_i,v_j): v_i.v_j \varepsilon V_0; i \neq j\}$ is the **arc set.** Every vertex of V corresponds to a city or customer; and vertex v_0 represents a depot at which exactly m or at most m vehicles are based. Depending on the model considered; m is either a constant or a decision variable representing the number of vehicles used in the optimal solution. or it is an upper bound on the number of available vehicles. This will be made explicit in the various models that follow. With E is associated a **distance** or **travel cost** matrix $C=(c_{ij})$. If C is symmetrical. as will be the case in this paper. E can be defined as a set of undirected **edges** $(v_i.v_j)$ where $i<j$. The VRP consists of determining a set of vehicle routes

(i) starting and ending at the depot. and such that

(ii) every vertex of V is visited exactly once by exactly one vehicle;

(iii) some side constraints on the vehicle routes are satisfied;

(iv) a linear combination of the number of vehicles and of the total routing cost is minimized.

Note that when C satisfies the triangle inequality (i.e. $c_{ij} \leq c_{ik} + c_{kj}$ for all i.j.k). condition (ii) is not restrictive. This occurs. for example. if c_{ij} is the length of a shortest path between i and j. There exists a wide variety of side constraints; for examples and recent surveys on the VRP. see Laporte and Nobert [14] and Golden and Assad [6].

This statement of the VRP does not adequately describe a variety of practical situations where one or several parameters are **stochastic.** This is the case. for example. where vehicles of fixed capacities must collect random quantities at the various customers (see for example Laporte. Louveaux and Mercure [12]. as well as Dror. Laporte and Trudeau

[5]). Another case is where customers are visited on a random basis (Jaillet [7] and Bertsimas [1]). In this paper, we assume stochastic travel times. More precisely, a stochastic travel time matrix $T=(t_{ij})$ is associated with E, and a stochastic service time vector $\tau=(\tau_i)$ is defined on V. Travel or service times are continuous or discrete random variables that may be independent or not. Examples of dependent travel times are encountered when traffic conditions are affected by the weather: on a rainy, foggy or snowy day, driving is likely to be more difficult on all segments of a road network. It then makes sense to envisage a limited number of random scenarios where all travel times are affected in the same manner. At the other extreme, independent travel times occur in situations where traffic may be delayed on a particular stretch of road (because of repairs, for example), but not on another. The models and the general algorithm developed in this paper apply in principle whether or not the random variables are independent, although problems are usually easier to solve exactly when the total number of states for the entire system is limited.

The problem under study in this paper is called the Vehicle Routing Problem with Stochastic Travel Times (abbreviated as SVRP). It consists of planning optimal vehicle routes in the presence of random travel and service times. Once routes are planned, the values taken by the random variables become known and it is possible to compute the actual duration of a route for these values of the random variables. Here, vehicles incur a penalty proportional to the duration of their route in excess of a preset constant B. Such problems occur frequently in contexts where drivers must be paid overtime for work done after normal hours. A variant of this problem is where the penalty for route duration in excess of B is proportional to the vehicle content at arrival. Such is the case when armoured vehicles are used to collect money from bank branches to a central bank: here, all transactions registered after a certain time of day are credited the next day, thus losing one day's interest (Lambert, Laporte and Louveaux [10]).

As is standard in stochastic programming (see, for example, Kall [8]), two versions of the problem will be considered:

(i) in chance constrained programming, the objective is to minimize planned route costs, while ensuring that the probability of having a route duration in excess of B does not exceed a given treshold;

(ii) in stochastic programming with recourse, the objective is to minimize the sum of planned route costs and of expected penalty costs.

Stochastic VRPs are considerably more difficult to solve than their deterministic counterparts and have thus received less attention from operations researchers. To the authors' knowledge, our problem has never been previously treated in the Operations Research literature. The only related studies appear to be Leipälä's paper [15] which considers the problem of estimating the length of the optimal tour in a traveling salesman problem with stochastic distances, and the papers by Kao [9], Sniedovich [16] and Carraway, Morin and Moskowitz [3], in which the objective is to determine a tour with maximum probability of completion by a specified time. Our objective is to provide workable formulations and exact algorithms for SVRPs of moderate sizes.

This paper is organized as follows. In Section 1, the SVRP is formulated as a chance constrained problem. Two recourse models for the same problem follow in Sections 2 and 3. A general branch and cut algorithm for the models of Sections 1, 2 and 3 is described in Section 4 while computational results are presented in Section 5 for moderate size problems involving a limited number of random states. The conclusion follows in Section 6.

1. CHANCE CONSTRAINED MODEL

The objective in the chance constrained model is to minimize a linear combination of vehicle and routing costs while ensuring that the probability of the duration of a route exceeding B is at most equal to a given value α. Define

x_{ij} $(i<j)$: the number of times edge $(v_i.v_j)$ is used in the optimal solution. This variable is equal to 0 or 1 if $0<i<j$ and to 0; 1 or 2 if $i=0$; the case $x_{0j}=2$ corresponds to a return trip between the depot and v_j;

$x = (x_{ij})$;

f : the fixed cost (per planning period) of a vehicle.

The model is then

$$(P1) \qquad \min_{x,m} \; fm + \sum_{i<j} c_{ij}\, x_{ij}.$$

subject to

$$\sum_{j=1}^{n} x_{0j} = 2m \tag{1}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \qquad (v_k \varepsilon V) \tag{2}$$

Illegal route elimination constraints (3)

$$x_{ij} \varepsilon \{0;1\} \qquad (v_i.v_j \varepsilon V) \tag{4}$$

$$x_{0j} \varepsilon \{0;1;2\} \qquad (v_j \varepsilon V) \tag{5}$$

$m \geq 1$ and integer. (6)

In this model, m can be regarded as a constant or as a decision variable. All constraints except (3) are self-explanatory. Illegal routes referred to in constraints (3) include subtours disconnected from the depot and vehicle routes whose probability of having a duration in excess of B is larger than α. As in the travelling salesman problem (TSP) (Dantzig, Fulkerson and Johnson [4]) illegal subtours can be eliminated by imposing the connectivity constraints

$$\sum_{\substack{v_i \in S, v_j \notin S \\ \text{or } v_i \notin S, v_j \in S}} x_{ij} \geq 2 \qquad (S \subseteq V,\ 3 \leq |S| \leq n-3). \tag{7}$$

Now consider a vehicle route $L = (v_{i_0} = v_0, v_{i_1}, \ldots, v_{i_u}, v_{i_0})$. Such a route is illegal if

$$P\left(\sum_{k=0}^{u} \left[t_{i_k i_{k+1}} + \tau_{i_k}\right] > B\right) > \alpha\ , \tag{8}$$

where $i_{u+1} = i_0$ and $\tau_{i_0} = 0$. In this case, L can be eliminated by imposing

$$\sum_{k=0}^{u} x_{i_k i_{k+1}} \leq u\ . \tag{9}$$

Note that this constraint only eliminates route L, but none of the routes obtained by permuting the components of L, since these may be legal. However, when the following assumption is satisfied, it is valid to eliminate these permutations by imposing a stronger constraint. This is done as follows. Let $S \subseteq V$ and let L_S be a permutation of the elements of $S \cup \{v_0\}$, corresponding to a vehicle route. Let also $t(L_S)$ be the total duration of L_S (including service time), $c(L_S)$ its total cost, and $L_S^* \in \text{argmin}\{c(L_S)\}$ be a least cost permutation of the cities of $S \cup \{v_0\}$ (under the current set of constraints).

Assumption 1

$$P(t(L_S) > s) \geq P(t(L_S^*) > s) \text{ for all } s \geq B\ . \qquad \blacksquare$$

Note that this assumption is weaker than the assumption of monotonicity of t with respect to c, i.e.

$$c(L_S) \geq c(L_S^*) \Rightarrow t(L_S) \geq t(L_S^*) \text{ a.s.}$$

Also observe that if travel times are deterministic and service times are the only source of uncertainty, then Assumption 1 reduces to $t(L_S) \geq t(L_S^*)$.

Proposition 1

If L_S^* satisfies (8) and Assumption 1 holds, then the following constraint is valid:

$$\sum_{\substack{v_i \in S, v_j \notin S \\ \text{or } v_i \notin S, v_j \in S}} x_{ij} \geq 2\, V_\alpha(S) \qquad (S \subseteq V)\,, \tag{10}$$

where $V_\alpha(S)$ is a lower bound on the number of vehicles required to visit v_0 and all cities of S, so that the probability of incurring a penalty in $S \cup \{v_0\}$ does not exceed α. It can be taken as the smallest integer satisfying

$$P(t(L_S^*) > B\, V_\alpha(S)) \leq \alpha\,. \tag{11}$$

Proof. By Assumption 1, if L_S^* is a route of least duration, there is no other permutation L_S such that (11) would be satisfied with a strictly smaller value of $V_\alpha(S)$. Hence, at least $V_\alpha(S)$ vehicles are required to visit v_0 and all cities of S, and twice as many edges must link S and its complement. This is expressed by contraints (10). Constraint (11) computes the probability that at least one vehicle will incur a penalty while visiting $S \cup \{v_0\}$. It compares $t(L^*_S)$, the minimum time required by all vehicles to visit $S \cup \{v_0\}$, to the total time $V_\alpha(S)$ vehicles can travel in $S \cup \{v_0\}$ before incurring a penalty. ∎

Here, two remarks are in order. First, conditions (8) and (11) require computing the probability of a sum of random variables. This poses no difficulty in practice as long as these variables have a finite support of small cardinality or, as is frequently the case, if travel and service times have additive probability distributions (e.g. when they are normally and independently distributed). Second, the value of $V_\alpha(S)$ in (10) and (11) is usually not known a priori, but is gradually revealed at the optimum of every subproblem within a branch and cut process, when the values of $t^*(L_S)$ become known for all sets S corresponding to vehicle routes (see Section 4).

This chance constrained model is attractive and relatively simple to solve. However; in order to take into account the expected cost related to excess route duration; it is necessary to resort to a recourse model. Two such models are presented in the following two sections.

2. A THREE-INDEX SIMPLE RECOURSE MODEL

Several recourse models can be defined for the SVRP. In each of these; first stage decisions are made about the number of vehicles and their routes. Random variables corresponding to stochastic travel and service times are then realized and, in the second stage, penalties are incurred for excess durations. We first describe a three-index model. Define

m : an upper bound on the number of vehicles;

$x_{ij\ell}\ (i<j)$: the number of times (0, 1 or 2) edge $(v_i.v_j)$ is traversed by vehicle ℓ in the optimal solution;

$x = (x_{ij\ell})$;

$$z_{k\ell} = \begin{cases} 1 & \text{if } v_k \text{ is visited by vehicle } \ell; \\ 0 & \text{otherwise} \end{cases}$$

$z = (z_{k\ell})$;

ξ : a vector of random variables corresponding to travel and service times. Each realization of ξ is called a "state of the world";

Ξ : the support of ξ; in this model, Ξ is assumed to be finite;

$y_\ell(\xi)$: the excess duration of route ℓ in state of the world ξ;

$c_{ij\ell}$: the travel cost of vehicle ℓ on edge $(v_i.v_j)$;

$t^{\xi}_{ij\ell}$: the travel time of vehicle ℓ on edge $(v_i.v_j)$ in state of the world ξ;

$\tau^{\xi}_{i\ell}$: the service time of vertex v_i by vehicle ℓ in state of the world ξ;

β_ℓ : the (positive) unit penalty cost for excess duration on the route travelled by vehicle ℓ;

f_ℓ : the fixed cost of vehicle ℓ;

B_ℓ : the maximum time for route ℓ over which a penalty is incurred.

The model is then

$$(P2)\quad \min_{x,z}\ \sum_{\ell=1}^{m} f_\ell\, z_{0\ell} + \sum_{\ell=1}^{m} \sum_{i<j} c_{ij\ell}\, x_{ij\ell} + E_\xi\Big(\sum_{\ell=1}^{m} B_\ell\, y_\ell(\xi)\Big)$$

subject to:

$$\sum_{\ell=1}^{m} z_{k\ell} = 1 \qquad (v_k \in V) \tag{12}$$

$$\sum_{j=1}^{n} x_{0j\ell} = 2\, z_{0\ell} \qquad (\ell=1,\ldots,m) \tag{13}$$

$$\sum_{i<k} x_{ik\ell} + \sum_{j>k} x_{kj\ell} = 2\, z_{k\ell} \qquad (v_k \in V;\ \ell=1,\ldots,m) \tag{14}$$

$$\sum_{\substack{i,j\in S\\ i<j}} x_{ij\ell} \leq |S| - 1 \qquad (S \subseteq V,\ 3 \leq |S| \leq n-3;\ \ell=1,\ldots,m) \tag{15}$$

$$B_\ell - \sum_{i<j} t^{\xi}_{ij\ell}\, x_{ij\ell} - \frac{1}{2} \sum_{i<j} (\tau^{\xi}_{i\ell} + \tau^{\xi}_{j\ell})\, x_{ij\ell} + y_\ell(\xi) \geq 0 \qquad (\ell=1,\ldots,m;\ \xi\in\Xi) \tag{16}$$

$$x_{ij\ell} \in \{0,1\} \qquad (v_i, v_j \in V;\ \ell=1,\ldots,m) \tag{17}$$

$$x_{0i\ell} \in \{0,1,2\} \qquad (v_j \in V;\ \ell=1,\ldots,m) \tag{18}$$

$$z_{k\ell} \in \{0,1\} \qquad (v_k \in V_0;\ \ell=1,\ldots,m) \tag{19}$$

$$v_\ell(\xi) \geq 0 \qquad (\ell=1,\ldots,m;\ \xi\in\Xi). \tag{20}$$

Constraints (12)-(15) and (17)-(20) of this model are standard or self-explanatory. In constraints (16), the second term computes the total travel time on route ℓ, whereas the third term corresponds to its total service time. A factor of ½ is used in order to avoid double counting of

service times. Constraints (16) combined with (20) imply that a penalty will be incurred by vehicle ℓ only if the duration of its route, including service time, exceeds B_ℓ.

The second stage problem is a recourse model with continuous decision variables. It follows that the second-stage expected optimal value function is a convex function of x and that it is piecewise linear in x since Ξ has a finite support. Also note that the second stage problem is a simple recourse model. Indeed, for a given ξ, this problem is separable in $v(\xi)=(v_1(\xi),\ldots,v_m(\xi))$ and can be written in the form

$$\text{minimize} \qquad q^+ y^+(\xi) + q^- y^-(\xi),$$

subject to

$$y^+(\xi) - y^-(\xi) = h(\xi) - T(\xi)x \qquad (\xi\varepsilon\Xi)$$

$$y^+(\xi),\ y^-(\xi)\geq 0 \qquad (\xi\varepsilon\Xi)$$

Here, $y^+(\xi)$ coincides with $y(\xi)$ and $q^- = 0$, so that $y^-(\xi)$ is not needed in the formulation, $h(\xi)= B = (B_1,\ldots,B_m)$ is independent of ξ, and $T(\xi)$ is the coefficient vector of x in (16).

3. A TWO-INDEX RECOURSE MODEL

We now present a two-index model for the SVRP. Here, variables x_{ij} indicate the absence or the presence of a trip on (v_i,v_j), without identifying which vehicle makes the trip, and $x=(x_{ij})$. This obviously results in an important saving in the number of first stage variables. On the other hand, the simple recourse property is lost since there is no linear continuous analytical expression that gives the penalty in terms of the first stage decision variable x. In this model, Ξ is not necesssarily finite and m can be a constant or a variable. However, it is now necessary

to assume all vehicles are identical, so that particular vehicle characteristics can no longer be taken into account. In addition to the variables already introduced in the previous two models, define

y : the total expected excess duration of a first stage solution x;

β : the (positive) unit cost of excess duration.

The model is then

$$\min_{x,m,y} \; fm + \sum_{i<j} c_{ij} x_{ij} + \beta y$$

subject to:

(1), (2), (4)-(7) and y is an unknown function of x.

We recall here that no linear continuous expression of y in terms of x exists, although one might suspect that combinatorial expressions could be derived. We circumvent this difficulty as follows. First observe that the SVRP has only a finite number of first stage solutions. Assume these are indexed by r (there is no need to specify the rule used to make the enumeration). For every r, define

$S_r = \{(v_i, v_j) : v_i, v_j \in V$ and (v_i, v_j) belongs to the r^{th} feasible SVRP solution$\}$;

Θ_r : a constant equal to the total expected excess duration of solution r.

Using this notation, the model can now be formally written as:

(P3) $$\min_{x,m,y} \; fm + \sum_{i<j} c_{ij} x_{ij} + \beta y$$

subject to:

(1), (2), (4)-(7) and

$$y \geq \Theta_r \left[\sum_{(v_i, v_j) \in S_r} x_{ij} - (n-m-1) \right] \qquad \text{(all r)} \qquad (21)$$

$$y \geq 0 . \qquad (22)$$

In this model, constraints (1), (2) and (4)-(7) provide a feasible solution for the m-TSP (see Laporte and Nobert [13]). Constraints (21) and (22), with the fact that y appears with a positive coefficient in the objective function, ensure that a penalty equal to Θ_r will be incurred if solution r is selected; otherwise, there is no constraint on y except (22). This can be explained as follows. If solution r is selected, then $|S_r|=n-m$, since each of the n vertices of V is entered through one edge, out of which the m edges incident to v_0 must be subtracted. Therefore, $\sum_{(v_i,v_j) \in S_r} x_{ij} = n-m$, since x_{ij} is a binary variable equal to 1 whenever $(v_i, v_j) \in S_r$. If, on the other hand, solution r is not selected, then $\sum_{(v_i,v_j) \in S_r} x_{ij} \leq n-m-1$. Constraints (21) simply state that the penalty y must lie above a straight line passing through (n-m-1, 0) and (n-m, Θ_r). The number of constraints (21) is very large, making (P3) potentially difficult to solve. As already mentioned, this difficulty comes from the absence of an expression or linear program defining y in terms of x in the two-index model. However, as shown in the next sections, these constraints are gradually introduced during a branch and cut process as they are found to be violated and our computational results confirm that very few of them are in fact generated. Finally, note that the computation of Θ_r poses no difficulty. It suffices to sum the expected excess duration on each route that constitutes the current solution. (These routes are easily identified when the x_{ij}'s are known). This in turn requires the knowledge of the distribution of the sum of travel and service times along the routes. Such a distribution is easily established when $|\Xi|$ is finite and small, or when travel and service times have additive probability distributions.

5. ALGORITHM

We briefly describe in this section a general branch and cut algorithm for the three models (P1), (P2) and (P3). A number of comments specific to each particular model then follow.

The constraints contained in these three models can be divided into five categories:

(i) **degree constraints** which specify the vehicles used at various vertices (12) and the degree of each vertex [(1),(2),(13),(14)];

(ii) **lower and upper bounds** on the variables, including constraints (20) and (22);

(iii) **integrality requirements** on the variables;

(iv) **illegal route and subtour elimination constraints** [(7),(9), (10), (15)];

(v) **lower bounds on penalties** for excess route durations [(16),(21)].

The algorithm for the three models can be outlined as follows:

Step 0. Let z^* be the cost of the best known feasible solution. If no solution is known, set $z^* := \infty$. Define a first current problem as the relaxed problem containing constraints (i) and (ii). Insert the current problem in a list.

Step 1. If the list is empty, print the best known solution and stop. Otherwise, select a problem from the list.

Step 2. Solve the current problem and let z be the value of its optimal solution. If $z > z^*$, fathom the current problem and go to step 1.

Step 3. If the solution is not integer, create subproblems by branching on a fractional variable, insert them in the list and go to step 1.

Step 4. At an integer solution, check first for subtours, and then for illegal routes. If any violation is detected, add the appropriate illegal route or subtour elimination constraints to the current problem and go to step 2.

Step 5. At a first stage feasible solution, let z' be the value of z, plus the value of expected penalty for excess duration. If $z > z^*$, fathom the current problem and go to step 1. If $z' < z^*$, update the best known solution and set $z^*:=z'$. Introduce in the current problem the appropriate type (v) constraint and go to step 2. ∎

The convergence of this branch and cut algorithm is guaranteed by the finite number of feasible first stage solutions, and by the finiteness of the number of (potential) constraints in each model.

In the chance constrained model (P1), constraints (v) and hence step 5 do not apply. Otherwise, the algorithm is identical to that of the deterministic distance constrained VRP (Laporte, Desrochers and Nobert [11]). When Assumption 1 is satisfied and constraints (10) are used, some difficulties may arise in the computation of $V_\alpha(S)$, since it involves solving a TSP on $S \cup \{v_0\}$. An easy case occurs when C satisfies the triangle inequality.

Proposition 2

Assume C satisfies the triangle inequality and consider an optimal solution to the current problem containing no subtours identified in step 4 of the algorithm. Let L_S be a route corresponding to this optimal solution. Then this solution is the least cost, way of visiting all vertices of S.

Proof. The current solution is feasible and optimal for the m-TSP at the current node of the search tree. Since C satisfies the triangle inequality, each vehicle route L_S taken separately is also optimal: (i) permuting or partitioning it cannot decrease the total distance required to serve $S \cup \{v_0\}$, (ii) moreover, sharing its vertices between several routes including vertices in $V \setminus S$ cannot yield a shorter distance since for any vertex sequence $(v_{i_1}, v_{i_2}, v_{i_3})$ on a route L with $v_{i_2} \notin S$, a route of equal or shorter distance can be obtained by deleting v_{i_2} and connecting v_{i_1} to v_{i_3}. Hence, there is no less costly way of visiting all vertices of $S \cup \{v_0\}$. ∎

Note that constraints (10) generated with this solution must be removed when backtracking, since the value of L_S is affected by the fixed x_{ij} variables at the current node of the search tree.

In the three-index recourse model (P2), whenever a type (v) constraint is generated for a given vehicle ℓ, it is also generated for all other vehicles in order to avoid the occurrence of an equivalent solution in which route indices are merely permuted. In this respect, the algorithm is similar to the multicut approach in two-stage stochastic programming (Birge and Louveaux [2]).

Finally, in the two-index recourse model (P3), constraints (21) provide a non-trivial lower bound on y only when the current first stage solution coincides with a first stage solution previously obtained. In order to gain more information on y, it is useful to introduce in the first problem considered at step 0, a more accurate lower bound on y. Such a bound is now provided.

Proposition 3

A valid lower bound on y is given by

$$y \geq E_{\xi}\Big(\sum_{i<j} t_{ij}^{\xi} x_{ij} + \frac{1}{2}\sum_{i<j}(\tau_i^{\xi}+\tau_j^{\xi})\, x_{ij} - mB\Big)^{+}, \qquad (23)$$

where $(x)^{+} = \max(x,0)$.

Proof. The validity of this bound rests on the fact that for any given ξ, the expression within parentheses compares an aggregate of all travel and service times with an aggregate of available times on each route. ∎

In practice, incorporating (23) into a linear program poses no difficulty when the number of states ξ is finite and relatively small. Let the states be indexed by $k=1,\ldots,|\Xi|$ and p^k represent their respective probabilities. Then (23) is equivalent to

$$v \geq \sum_{k=1}^{|\Xi|} p^k v^k \qquad (24)$$

$$y^k \geq \sum_{i<j} t_{ij}^k x_{ij} + \frac{1}{2} \sum_{i<j} (\tau_i^k + \tau_j^k) x_{ij} - mB \qquad (k=1,\ldots,|\Xi|) \quad (25)$$

and

$$y^k \geq 0 \qquad (k=1,\ldots,|\Xi|). \quad (26)$$

These constraints were used in the implementation of our algorithm for (P3).

When Ξ is infinite, or finite with a large cardinality, we can use the following weaker form of (23) since $x^+ \geq x$ always holds:

$$y \geq E_\xi \left[\sum_{i<j} t_{ij}^\xi x_{ij} + \frac{1}{2} \sum_{i<j} (t_i^\xi + t_j^\xi) x_{ij} \right] - mB.$$

This bound only requires the expectations of travel and service times and is easy to compute when these random variables are independent.

5. COMPUTATIONAL RESULTS

Of the three models and algorithms described in this paper, only the last two were tested. The chance constrained (P1) model is indeed very similar to the model developed for deterministic distance constrained VRPs (Laporte, Desrochers and Nobert [11]). The behaviour of such models is known and well documented (see Laporte and Nobert [14] for deterministic VRPs, and Laporte, Louveaux and Mercure [12] for the VRP with stochastic supplies).

Models (P2) and (P3) were tested on problems involving a limited number of random scenarios. Distance matrices (c_{ij}) were obtained by first generating points (X_i, X_j) in $[0,100]^2$ according to a uniform distribution,

and by then setting $c_{ij} = [(X_i - X_j)^2 + (Y_i - Y_j)^2]$. The number of states varied between 2 and 5. The value of f was equal to zero in all problems, and m was assumed to be a constant. Travel times $t_{ij\ell}$ were equal to $\xi c_{ij}/60$ for all ℓ and for $\xi=1,2,\ldots|\Xi|$; no service times were considered. The value of B was set equal to 6. If distances are expressed in kilometers and time units in hours, then these parameters translate into travel speeds of $60/\xi$ km/h on all edges, and penalties for trips in excess of 6 hours. The value of β, the unit penalty cost for excess duration, was allowed to vary between 5 and 20. This again can be compared with an average cost of one unit per kilometer.

Three problems were attempted for n=10, 15, 20 and for each combination of m and the number of states. All tests were run on the Université de Montréal CYBER 855 computer, with a maximum running time of 300 seconds per problem.

Preliminary tests quickly revealed the superiority of (P3) over (P2). The main difficulty with (P2), it seems, lies in the large integer programs associated with this model, even for relatively modest values of n. Extensive testing of this model was therefore abandoned in favour of (P3).

In order to test the two-index recourse model (P3), we attempted three problems for various combinations of the parameters considered. The algorithm "failed" when the problem could not be solved to optimality within 300 seconds or when memory capacity ($250\ 000_8$ words) became exceeded. Failures can be attributed to lack of time or lack of space in approximately equal proportions. Table 1 reports average results over problems for which the algorithm was successful. The meanings of the column headings which have not yet been defined are:

STATES : number of stochastic states (i.e. the cardinality of Ξ);

SUCC : number of successful problems out of 3;

PIVOTS : total number of simplex pivots in the search tree;

CONNEC : total number of connectivity constraints generated during the branch and cut process;

PENALTY : total number of type (21) penalty constraints generated;

LEVEL : maximum level of the search tree;

TIME : CPU time in seconds (although problems were run on a CYBER 855 computer, all times are CYBER 173 equivalent).

Table 1 : Computational results for the two-index recourse model

β	n	m	STATES	SUCC	PIVOTS	CONNEC	PENALTY	LEVEL	TIME
5	10	2	2	3	23.0	1.0	1.0	0.3	0.23
		2	3	3	51.0	2.3	2.0	2.0	0.70
		2	4	3	54.3	2.0	1.7	4.0	0.88
		2	5	3	64.7	2.0	2.3	2.0	1.23
		3	2	3	23.3	1.0	1.3	0.7	0.24
		3	3	3	28.7	0.7	1.3	1.3	0.35
		3	4	3	57.3	2.0	2.7	3.7	0.88
		3	5	3	45.3	2.0	3.3	2.7	0.96
	15	2	2	2	356.5	3.5	15.5	14.5	11.95
		2	3	3	444.3	3.7	9.7	14.0	16.51
		2	4	3	328.7	3.7	7.7	14.0	12.88
		2	5	3	359.0	5.7	7.3	14.3	15.62
		3	2	1	219.0	4.0	9.0	11.0	6.47
		3	3	2	327.0	3.5	9.5	14.5	11.33
		3	4	3	433.7	3.7	13.0	12.3	15.74
		3	5	3	366.3	3.7	9.3	14.3	16.62
	20	2	2	2	339.0	4.0	9.0	9.5	16.68
		2	3	3	415.7	6.3	8.7	11.7	25.96
		2	4	2	1398.5	3.0	16.0	25.0	103.04
		2	5	1	204.0	2.0	1.0	16.5	14.86
		3	2	2	415.5	2.5	12.0	14.0	19.39
		3	3	3	383.0	2.7	11.7	11.3	22.67
		3	4	2	860.5	7.0	20.5	19.5	61.98
		3	5	1	299.0	2.0	7.0	19.0	27.72
10	10	2	2	3	34.3	1.3	1.3	1.3	0.42
		2	3	3	82.3	2.7	3.7	4.3	1.38
		2	4	3	104.7	2.3	3.7	6.7	1.95
		2	5	3	134.3	2.3	6.3	5.7	3.29
		3	2	3	35.0	1.0	2.3	1.3	0.45
		3	3	3	37.7	1.3	2.3	2.3	0.51
		3	4	3	208.3	2.7	11.0	9.7	4.39
		3	5	3	110.3	2.7	9.0	4.7	2.90
	15	2	2	1	2390.0	4.0	58.0	20.0	98.98
		2	3	2	1381.0	4.0	24.0	20.0	58.16
		2	4	3	1646.0	6.0	32.7	21.0	79.86
		2	5	3	1254.3	7.0	24.3	26.0	64.24
		3	2	0	---	---	---	---	---
		3	3	1	258.0	4.0	10.0	13.0	9.14
		3	4	2	551.5	4.0	19.5	14.5	24.88
		3	5	3	724.0	7.0	17.0	14.5	30.76
	20	2	2	1	750.0	3.0	17.0	16.0	37.22
		2	3	1	862.0	2.0	11.0	22.0	51.22
		2	4	1	1000.0	2.0	11.0	27.0	67.79
		2	5	1	751.0	2.0	11.0	27.0	60.49
		3	2	0	---	---	---	---	---
		3	3	3	2039.3	5.6	37.3	21.7	138.45
		3	4	1	2540.0	4.0	48.0	23.0	185.52
		3	5	1	2446.0	4.0	47.0	27.0	204.05

Table 1 : Computational results for the two-index recourse model (cont'd)

β	n	m	STATES	SUCC	PIVOTS	CONNEC	PENALTY	LEVEL	TIME
15	10	2	2	3	71.3	2.0	3.0	3.3	0.99
		2	3	3	133.3	2.7	5.3	5.7	2.37
		2	4	2	131.5	2.0	5.5	8.0	2.58
		2	5	3	227.0	2.7	8.7	7.0	5.51
		3	2	2	91.0	1.5	10.5	6.0	1.20
		3	3	3	51.7	1.3	2.7	3.0	0.75
		3	4	2	231.5	2.0	10.5	8.5	5.09
		3	5	3	208.3	2.7	15.0	7.0	6.05
	15	2	2	0	---	---	---	---	---
		2	3	1	545.0	6.0	14.0	14.0	18.70
		2	4	1	945.0	6.0	21.0	22.0	37.34
		2	5	2	1401.0	7.0	31.0	23.0	67.51
		3	2	0	---	---	---	---	---
		3	3	1	592.0	4.0	20.0	12.0	19.70
		3	4	0	---	---	---	---	---
		3	5	2	1727.5	7.5	36.5	22.0	76.90
	20	2	2	1	2814.0	4.0	62.0	21.0	179.93
		2	3	1	2943.0	4.0	49.0	25.0	198.35
		2	4	1	3035.0	4.0	43.0	32.0	234.24
		2	5	1	2823.0	4.0	43.0	34.0	249.15
		3	2	0	---	---	---	---	---
		3	3	1	1212.0	6.0	16.0	16.0	73.31
		3	4	0	---	---	---	---	---
		3	5	0	---	---	---	---	---
20	10	2	2	3	107.3	2.3	4.0	4.7	1.59
		2	3	3	200.0	2.7	6.0	8.0	3.49
		2	4	3	334.3	2.3	10.0	10.0	6.75
		2	5	3	290.7	3.0	13.0	8.0	7.50
		3	2	2	21.0	1.5	0.5	0.0	0.19
		3	3	3	76.0	1.7	3.3	5.3	1.15
		3	4	2	338.0	2.0	14.0	10.0	7.85
		3	5	3	379.7	3.3	21.7	7.0	11.82
	15	2	2	0	---	---	---	---	---
		2	3	1	1359.0	6.0	30.0	17.0	49.91
		2	4	1	2253.0	6.0	41.0	25.0	98.56
		2	5	2	2971.0	7.0	47.0	31.5	153.69
		3	2	0	---	---	---	---	---
		3	3	1	932.0	5.0	35.0	20.0	35.67
		3	4	0	---	---	---	---	---
		3	5	0	---	---	---	---	---

Our results indicate that problem difficulty increases with β. This was to be expected since a large value of β requires the generation of more penalty constraints, which in turns leads to more pivots and larger CPU times. At the other extreme, setting β at a value close to 0 would have made the problem almost as easy as an m-TSP. Problem difficulty is only slightly affected by the number of states which determines the number of type (25) constraints. The number of vehicles used, on the other hand, seems to have no effect on the performance of the algorithm. Problems involving a large number of cities are of course harder to solve, mainly because they involve more integer variables and require more branching; this is reflected by the values of the LEVEL column. The small number of connectivity and penalty constraints generated in most problems indicates that our relaxation approach was quite effective.

6. CONCLUSION

We have modeled and solved in this paper a difficult class of stochastic VRPs arising in a number of practical situations. The chance constrained model is similar to those developed for deterministic VRPs. Two recourse models were presented: one uses three-index variables and explicitly lists all penalty constraints; the second recourse model contains two-index variables and can be solved by means of a constraint relaxation algorithm. Computational results indicate that it can be used for the exact solution of moderate size problems. Problem difficulty seems to be affected by the size of the penalty factor, but not so much by the number of vehicles or by the number of random scenarios.

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